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Symmetries in Taxicab Ellipse

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Abstract: This article presents geometric aspects of the ellipse analyzed from the point of view of the taxicab metric. We first construct the taxicab ellipse by exploring its symmetric properties. From this, we analyze the action of isometries of the Euclidean context and verify whether these are transformations that preserve distances on the taxicab geometry. In particular, we study the behaviour of the taxicab ellipse with respect to rotations. We also obtain an algebraic equation for the regular octagon in the plane directly from the fact that every regular octagon is an taxicab ellipse.
Keywords: Non-Euclidean geometry; Symmetries; Taxicab ellipse; Regular octagon

1. Introduction

The Euclidean metric is defined as the measure of a line segment connecting any two points, derived from the basic geometry developed by Euclid (300 b.C.). This method of measurement is not always the best choice to represent the displacement of people and vehicles, particularly in large urban centers, because moving in a straight line is not always an option. Thus, choosing a more convenient metric is a good way to describe the urban displacement and, in this sense, the taxicab metric is a good alternative.

The taxicab geometry was introduced by the German mathematician Hermann Minkowski (1864-1909). It is a form of geometry in which the Euclidean metric is replaced by the sum metric, or the so-called taxicab metric, given on the Euclidean plane by the function $d_T : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$d_T((x_1, y_1), (x_0, y_0)) = |x_1 - x_0| + |y_1 - y_0|. \tag{1}$$

This distance gives the minimum length of a path from $(x_1, y_1) \in \mathbb{R}^2$ to $(x_0, y_0) \in \mathbb{R}^2$ constructed from horizontal and vertical line segments (see Figure 1). So the taxicab metric, also called Manhattan metric, is the most used in urban networks because it is more consistent with the perpendicular lines of the streets and avenues of modern and planned cities. We cite references (DREILING, 2012; HANSON, 2012; INY, 1984; KRAUSE, 1986; KUNWAR, 2018; REINHARDT, 2005; SOWELL, 1989; THOMPSON; DRAY, 2000) for an introductory study of concepts of taxicab geometry. In these works, the authors explore this new geometry in relation to Euclidean geometry. In (REINHARDT, 2005), for example, the distance defined in (1) is used to find the solutions to three problems proposed by Eugene F. Krause in (KRAUSE, 1986).

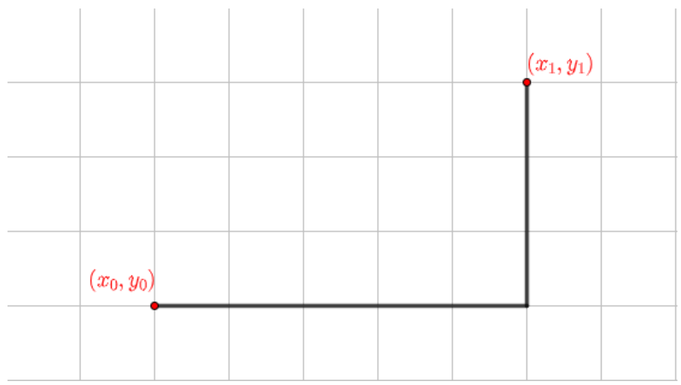

Figure 1: Taxicab metric.

Therefore, taxicab geometry is a non-Euclidean geometry with the advantage of being quite intuitive compared to other non-Euclidean geometries, although some geometric properties obtained in the usual metric do not remain in this new perspective. An example of this is the change in the geometric configuration of the conics, with the appearance of singularities. In this direction, taxicab geometry reveal interesting and surprising properties, as can be seen in (CHICIU, 2012; CRUZ, 2015; LOIOLA; COSTA, 2015; PETROVIC et al., 2025). In particular, in (CHICIU, 2012) the author examines conics under different metrics, including the taxicab metric, by a graphical approach. In (CRUZ, 2015), it is presented some symmetry conditions of taxicab ellipse that help in the recognition of its geometric properties. In (PETROVIC et al., 2025), the authors analyze the geometry of some curves (conics, circles and trifocal ellipses) in this context, including the study of area and perimeter.

In this article, we analyze the ellipse defined under the taxicab metric by taking advantage of its symmetries in the process of its geometric construction. Section 2 contains general results for this construction. In Section 3 we verify if Euclidean isometries also preserve distances in this new context. In Section 4 we use the taxicab ellipse to provide an algebraic equation for the set of points in a regular octagon.

2. Ellipse on the taxicab geometry

We start with the definition of an ellipse in the Euclidean plane $\mathbb{R}^2$.

Definition 2.1 *Let $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be an arbitrary metric and $F_1, F_2 \in \mathbb{R}^2$ two fixed points such that $d(F_1, F_2) = 2c$, for some $c > 0$. An ellipse Φ_{F_1,F_2} of foci F_1 and F_2 is the locus of points on the plane whose sum of the distances to F_1 and to F_2 is a constant $2a > 0$, where $a > c$. Symbolically,*

$$\Phi_{F_1,F_2} = \{P \in \mathbb{R}^2 : d(P, F_1) + d(P, F_2) = 2a\}.$$

Therefore, an ellipse is a plane curve surrounding two focus points F_1 and F_2. The standard form of an ellipse in the Euclidean metric is known as in Figure 2. However, the metric d in the Definition 2.1 is arbitrary and we can consider the taxicab metric.

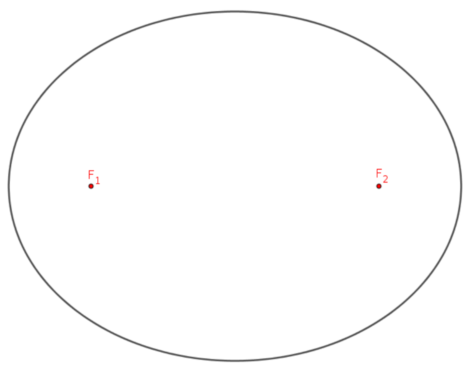

Figure 2: Euclidean ellipse.

From now on, we consider the Euclidean plane $\mathbb{R}^2$ equipped with the taxicab metric defined in (1), namely

$$d_T(P, Q) = |x_1 - x_0| + |y_1 - y_0|, \tag{2}$$

for all $P = (x_1, y_1),\ Q = (x_0, y_0) \in \mathbb{R}^2$. We will call the ellipse defined with the metric d_T of taxicab ellipse, or simply of taxi-ellipse. According to Definition 2.1, if $F_1 = (x_1, y_1)$ and $F_2 = (x_2, y_2)$ are the foci of the taxi-ellipse, then Φ_{F_1,F_2} is the set of all points $(x, y) \in \mathbb{R}^2$ such that

$$|x - x_1| + |y - y_1| + |x - x_2| + |y - y_2| = 2a. \tag{3}$$

In the search for ordered pairs that satisfy equation (3), we recognize the importance of investigating some type of symmetry in the taxi-ellipse. For this purpose, we present the following proposition.

Proposition 2.2 *Let Φ_{F_1,F_2} be a taxi-ellipse of foci $F_1 = (x_1, y_1)$ and $F_2 = (x_2, y_2)$. Then $(x, y) \in \Phi_{F_1,F_2}$ if and only if $(x, y_1 + y_2 - y) \in \Phi_{F_1,F_2}$. Similarly, $(x, y) \in \Phi_{F_1,F_2}$ if and only if $(x_1 + x_2 - x, y) \in \Phi_{F_1,F_2}$.*

Proof: We have $(x, y) \in \Phi_{F_1,F_2}$ if and only if (x, y) satisfies the equation (3). Obviously, this equality is the same as

$$|x - x_1| + |y_2 - y| + |x - x_2| + |y_1 - y| = 2a,$$

which is equivalent to

$$|x - x_1| + |(y_1 + y_2 - y) - y_1| + |x - x_2| + |(y_1 + y_2 - y) - y_2| = 2a.$$

In turn, this last equality occurs if and only if $(x, y_1 + y_2 - y) \in \Phi_{F_1,F_2}$. In the same way, (3) is the same as

$$|x_2 - x| + |y - y_1| + |x_1 - x| + |y - y_2| = 2a,$$

which is equivalent to

$$|(x_1 + x_2 - x) - x_1| + |y - y_1| + |(x_1 + x_2 - x) - x_2| + |y - y_2| = 2a.$$

Therefore, $(x, y) \in \Phi_{F_1,F_2}$ if and only if $(x_1 + x_2 - x, y) \in \Phi_{F_1,F_2}$.

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Corollary 2.3 *Let Φ_{F_1,F_2} be a taxi-ellipse of foci $F_1 = (x_1, y_1)$ and $F_2 = (x_2, y_2)$. Then $(x, y) \in \Phi_{F_1,F_2}$ if and only if $(x_1 + x_2 - x, y_1 + y_2 - y) \in \Phi_{F_1,F_2}$.*

Using Proposition 2.2 and Corollary 2.3, we can do an alternative analysis of the geometric behavior of a taxi-ellipse taking into account its symmetries. For this, assume without loss of generality that $x_1 \le x_2$ and $y_1 \le y_2$. Given $(x, y) \in \Phi_{F_1,F_2}$, consider the cases:

(i) If $x \le x_1$ and $y \le y_1$, the equation (3) reduces to

$$-x + x_1 - y + y_1 - x + x_2 - y + y_2 = 2a,$$

that is, (x, y) lies on the straight line

$$r: \; y = -x - a + \frac{1}{2}(x_1 + y_1 + x_2 + y_2).$$

(ii) If $x \leq x_1$ and $y_1 \leq y \leq y_2$, the equation (3) becomes

$$-x + x_1 + y - y_1 - x + x_2 - y + y_2 = 2a,$$

that is, (x, y) lies on the straight line

$$s: \ x = -a + \frac{1}{2}(x_1 - y_1 + x_2 + y_2).$$

(iii) If $x_1 \leq x \leq x_2$ and $y \leq y_1$, the equation (3) becomes

$$x - x_1 - y + y_1 - x + x_2 - y + y_2 = 2a,$$

that is, (x, y) lies on the straight line

$$t: \ y = -a + \frac{1}{2}(-x_1 + y_1 + x_2 + y_2).$$

The geometric representation of the points $(x, y) \in \Phi_{F_1,F_2}$ that satisfy the cases (i), (ii) and (iii) is given in Figure 3, for hypothetical F_1 and F_2.

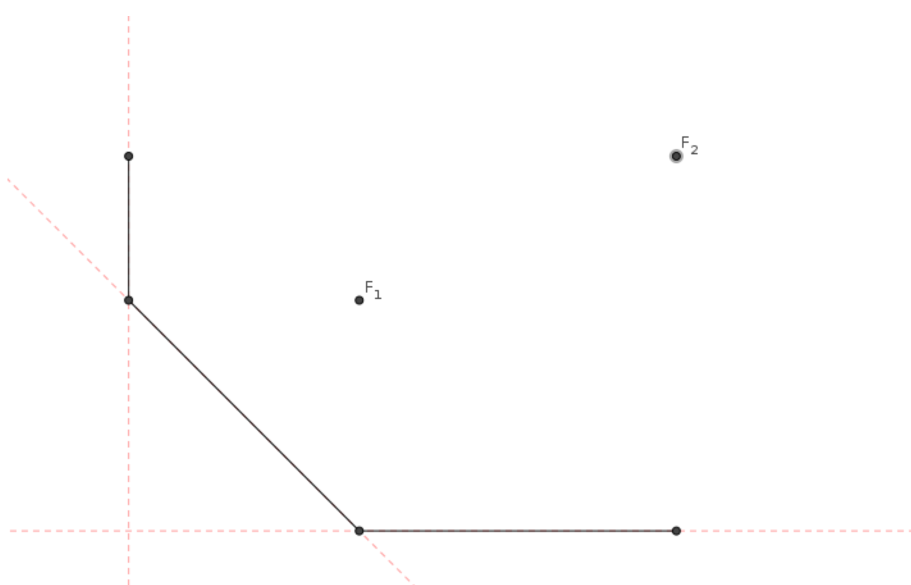

Figure 3: Part of a taxi-ellipse.

There are more straight lines in the plane that satisfy equation (3). In fact, by Proposition 2.2, if $(x, y) \in \Phi_{F_1,F_2}$, then $(x_1 + x_2 - x, y) \in \Phi_{F_1,F_2}$. Supposing that $x \geq x_2$ and $y \leq y_1$, we have

$$x' = x_1 + x_2 - x \leq x_1 \quad \text{and} \quad y \leq y_1$$

In this case, since $(x', y) \in \Phi_{F_1,F_2}$, it follows that $(x', y) \in r$, that is,

$$y = -x' - a + \frac{1}{2}(x_1 + y_1 + x_2 + y_2).$$

Therefore, (x, y) lies on the line

$$r' : \; y = x - a + \frac{1}{2}(-x_1 + y_1 - x_2 + y_2).$$

In other words, for $x \geq x_2$ and $y \leq y_1$, the points (x, y) that belong to the taxicab ellipse Φ_{F_1,F_2} are on the line segment r'.

Similarly, if $(x, y) \in \Phi_{F_1,F_2}$, then $(x, y') \in \Phi_{F_1,F_2}$, where $y' = y_1 + y_2 - y$. Assuming that $x \leq x_1$ and $y \geq y_2$, we have $x \leq x_1$ and $y' \leq y_1$. In this case, $(x, y') \in r$, that is,

$$y' = -x - a + \frac{1}{2}(x_1 + y_1 + x_2 + y_2).$$

Thus, (x, y) lies on the line

$$s' : \; y = x + a + \frac{1}{2}(-x_1 + y_1 - x_2 + y_2).$$

Therefore, for $x \leq x_1$ and $y \geq y_2$ the points $(x, y) \in \Phi_{F_1,F_2}$ are on the line segment s'.

Now using Corollary 2.3, if $(x, y) \in \Phi_{F_1,F_2}$, then $(x', y') \in \Phi_{F_1,F_2}$, where $x' = x_1 + x_2 - x$ and $y' = y_1 + y_2 - y$. Note that $x \geq x_2$ and $y \geq y_2$ if and only if $x' \leq x_1$ and $y' \leq y_1$. In this case $(x', y') \in r$, that is,

$$y' = -x' - a + \frac{1}{2}(x_1 + y_1 + x_2 + y_2),$$

so that (x, y) lies on the line

$$t' : \; y = -x + a + \frac{1}{2}(x_1 + y_1 + x_2 + y_2).$$

Therefore, for $x \geq x_2$ and $y \geq y_2$, the points $(x, y) \in \Phi_{F_1,F_2}$ are on the line segment t'.

So far, we obtain the taxi-ellipse for the regions of the plane such that $x \leq x_1$; $x_1 \leq x \leq x_2$ and $y \leq y_1$; $x \geq x_2$ and $y \leq y_1$; $x \geq x_2$ and $y \geq y_2$. The geometric representation of Φ_{F_1,F_2} is in accordance with Figure 4.

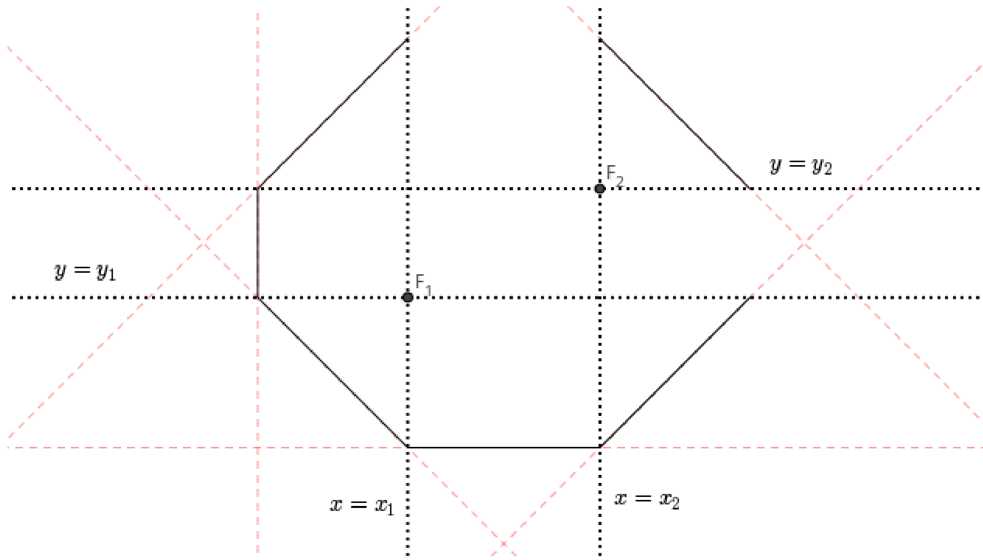

Figure 4: Incomplete taxi-ellipse.

Only three cases remain to be analyzed: $x \geq x_2$ and $y_1 \leq y \leq y_2$; $x_1 \leq x \leq x_2$ and $y \geq y_2$; $x_1 \leq x \leq x_2$ and $y_1 \leq y \leq y_2$. First, we assume $x \geq x_2$ and $y_1 \leq y \leq y_2$. Again, if $(x, y) \in \Phi_{F_1,F_2}$, then $(x', y) \in \Phi_{F_1,F_2}$, where $x' = x_1 + x_2 - x$. In this case $x' \leq x_1$ and $y_1 \leq y \leq y_2$, so that $(x', y) \in s$. Therefore

$$x' = -a + \frac{1}{2}(x_1 - y_1 + x_2 + y_2),$$

which implies that (x, y) belongs to the line

$$u: \; x = a + \frac{1}{2}(x_1 + y_1 + x_2 - y_2).$$

We assume now that $x_1 \leq x \leq x_2$ and $y \geq y_2$. Again, if $(x, y) \in \Phi_{F_1,F_2}$, then $(x, y') \in \Phi_{F_1,F_2}$, where $y' = y_1 + y_2 - y$. In this case $x_1 \leq x \leq x_2$ and $y' \leq y_1$, so that $(x, y') \in t$. Thus,

$$y' = -a + \frac{1}{2}(-x_1 + y_1 + x_2 + y_2),$$

or equivalently

$$u': \; y = a + \frac{1}{2}(x_1 + y_1 - x_2 + y_2).$$

Therefore, for $x_1 \leq x \leq x_2$ and $y \geq y_2$, the only points $(x, y) \in \Phi_{F_1,F_2}$ are on the line segment u'. It remains only to analyze the case $x_1 \leq x \leq x_2$ and $y_1 \leq y \leq y_2$. For this region, the equation (3) becomes

$$x_2 - x_1 + y_2 - y_1 = 2a.$$

Then

$$d_T(F_1, F_2) = |x_2 - x_1| + |y_2 - y_1| = 2a.$$

Since $d_T(F_1, F_2) = 2c$, we conclude that $c = a$, which contradicts the hypothesis $a > c$ in Definition 2.1. Therefore, there are no points in the taxicab ellipse Φ_{F_1,F_2} in this region.

Note that the geometric configuration of Φ_{F_1,F_2} for the case $x_1 \leq x_2$ and $y_1 \leq y_2$ is that of an octagon, as shown in Figure 5. However, we also have taxi-ellipses in the form of hexagons, as we shall see in the next section. We suggest using GeoGebra software (GEOGEBRA. . . , 2025) for a better visualization of the geometrical behaviour of taxi-ellipses when we vary the foci F_1 and F_2.

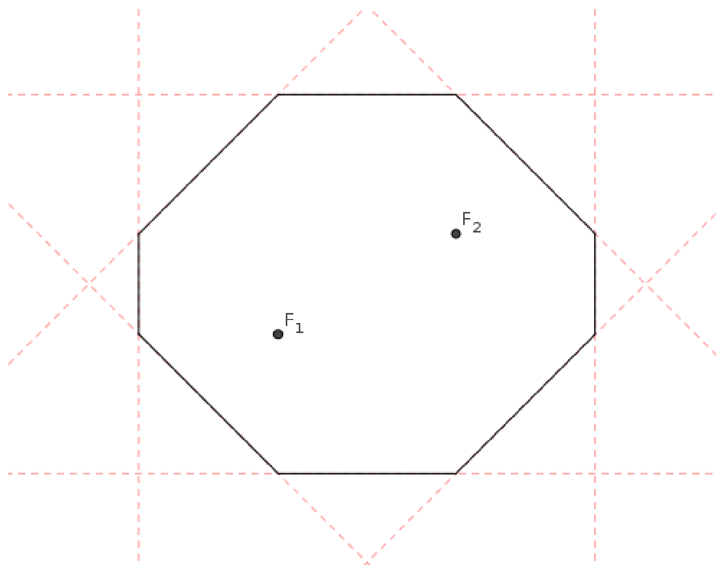

Figure 5: Taxi-ellipse.

3. Symmetries of the taxi-ellipse

In this section, we present some properties of the taxi-ellipse related to its symmetries in the Euclidean context. Geometrically, symmetries are defined in terms of isometries, that is, geometric transformations that preserve the distance between any two points. However, not every isometry is a symmetry. More specifically, we have the following definition.

Definition 3.1 *Let (M, d_M) and (N, d_N) be metric spaces. We say that $f : M \to N$ is an isometry if $d_N(f(x), f(y)) = d_M(x, y)$, for all $x, y \in M$.*

With the standard metric, isometries are transformations that do not change the size and the configuration of a figure, but they can change its position on the space. In this context, we cite the translations, rotations, reflections and glide reflections as the only four types of isometries (different from the identity) on the plane. Based on the concept of isometry, we define the symmetries of a figure as follows.

Definition 3.2 *We say that a plane figure is symmetric if there exists an isometry on* $\mathbb{R}^2$ *different from the identity that transforms that figure into itself. In this case, such isometry is called a symmetry of the figure.*

Therefore, by the geometric configuration of the taxi-ellipse Φ_{F_1,F_2} obtained in the Figure 5, we can see that some rotations and reflections behave as symmetries of Φ_{F_1,F_2} in the Euclidean context. For example, we will show in the next section that every regular octagon is a taxi-ellipse. In this case, rotation around the origin of angle $\theta = \frac{\pi}{4}$, as well as reflections around coordinate axes, constitutes its Euclidean symmetries.

It is important to question whether rotations and reflections also act as symmetries in taxicab geometry. To address this, we first verify whether translations, rotations, and reflections are isometries under the metric d_T defined in (2). We will see that the answer to this question is affirmative for translations and negative for rotations and reflections.

Proposition 3.3 *Translations under the taxicab metric are isometries.*

Proof: Let $T_a : \mathbb{R}^2 \to \mathbb{R}^2$ be the translation $T_a(x, y) = (x + a_1, y + a_2)$, with $a = (a_1, a_2) \in \mathbb{R}^2$ fixed. For any $(x, y), (x_1, y_1) \in \mathbb{R}^2$, we have

$$\begin{aligned} d_T((x,y),(x_1,y_1)) &= |x - x_1| + |y - y_1| \\ &= |(x + a_1) - (x_1 + a_1)| + |(y + a_2) - (y_1 + a_2)| \\ &= d_T((x + a_1, y + a_2), (x_1 + a_1, y_1 + a_2)) \\ &= d_T(T_a(x, y), T_a(x_1, y_1)), \end{aligned}$$

implying that every translation is an isometry in the taxicab geometry.

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Also in the Euclidean plane, a rotation is a transformation that turns every point of a figure through a specified angle and around a fixed point. The rotation around the origin of angle $\theta \in \mathbb{R}$ is the linear mapping $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$R_\theta(x, y) = (\cos\theta\, x - \sin\theta\, y, \sin\theta\, x + \cos\theta\, y).$$

As we mentioned, R_θ is an isometry under the Euclidean metric for all $\theta \in \mathbb{R}$. However, the same is not true under taxicab metric. In fact, consider $P = (1, 0)$ and $Q = (0, 1)$. Then

$$d_T(P, Q) = |1 - 0| + |0 - 1| = 2$$

and

$$d_T(R_\theta(P), R_\theta(Q)) = d_T((\cos\theta, \sin\theta), (-\sin\theta, \cos\theta)) = |\cos\theta + \sin\theta| + |\sin\theta - \cos\theta|.$$

For $0 < \theta \leq \dfrac{\pi}{4}$, we have

$$d_T(R_\theta(P), R_\theta(Q)) = \cos\theta + \sin\theta - \sin\theta + \cos\theta = 2\cos\theta \neq 2.$$

Similar calculations show that $d_T(R_\theta(P), R_\theta(Q)) \neq d_T(P,Q)$, for all $\theta \neq \frac{k\pi}{2}$, with $k \in \mathbb{Z}$. It is important to emphasize that R_θ is an isometry under the taxicab metric for all $\theta = \frac{k\pi}{2}$, with $k \in \mathbb{Z}$. For the simplicity of the computations, we omit the proof of this statement here.

Similarly, reflections can change the distance between two points of $\mathbb{R}^2$ in the taxicab geometry. More precisely, a reflection is a transformation whose set of fixed points forms a hyperplane. In two dimensions, this set is called the axis of reflection, and the image of a figure under a reflection is its mirror image across this axis. In algebraic terms, a reflection across a straight line that passes through the origin and forms an angle α with the x-axis, for $0 \leq \alpha < \pi$, is the linear mapping $M_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$M_\alpha(x, y) = (\cos 2\alpha\, x + \sin 2\alpha\, y, \sin 2\alpha\, x - \cos 2\alpha\, y).$$

For $P = (1, 0)$, $Q = (0, 1)$ and $0 < \alpha \leq \dfrac{\pi}{8}$, we have

$$\begin{aligned} d_T(M_\alpha(P), M_\alpha(Q)) &= d_T((\cos 2\alpha, \sin 2\alpha), (\sin 2\alpha, -\cos 2\alpha)) \\ &= |\cos 2\alpha - \sin 2\alpha| + |\sin 2\alpha + \cos 2\alpha| \\ &= \cos 2\alpha - \sin 2\alpha + \sin 2\alpha + \cos 2\alpha \\ &= 2\cos 2\alpha \neq 2. \end{aligned}$$

Direct calculations show that $d_T(M_\alpha(P), M_\alpha(Q)) \neq d_T(P,Q)$, for all $\alpha \neq \frac{k\pi}{4}$, with $k \in \{0, 1, 2, 3\}$. However, M_α is an isometry under the taxicab metric for all $\alpha = \frac{k\pi}{4}$, with $k \in \{0, 1, 2, 3\}$. Again, we omit the proof of this statement. Therefore, we have the following result:

Proposition 3.4 *Rotations R_θ are isometries under the taxicab metric if and only if $\theta = \frac{k\pi}{2}$, with $k \in \mathbb{Z}$. In the same way, reflections M_α are isometries under the taxicab metric if and only if $\alpha = \frac{k\pi}{4}$, with $k \in \{0, 1, 2, 3\}$.*

We then conclude that in the taxicab geometry the rotation of angle $\theta = \frac{\pi}{4}$ can not be a symmetry of the regular octagon, contrary to what occurs in the Euclidean context. The previous proposition suggest that rotations and reflections can change the size and shape of a plane figure in this new context. To verify this, we analyze the behavior of the taxi-ellipse under rotations by applying usual trigonometric relations (of Euclidean geometry). For an approach to trigonometry developed in the taxicab geometry, we refer to (THOMPSON; DRAY, 2000).

For the remainder of this section we follow the approach presented in (CRUZ, 2015, pp. 42-48), where the author discusses the rotation movements of the focal line of a taxi-ellipse for the particular case in which $c = 1$ and $a = 2$. In our study, the values of c and a ($a > c$) are arbitrary.

Consider the foci $F_1 = (x_1, y_1)$ and $F_2 = (x_2, y_2)$ of the taxi-ellipse Φ_{F_1,F_2} with respect to the canonical basis of $\mathbb{R}^2$. Denote by $\hat{r}$ the focal line determined by F_1 and F_2. We want to write the foci F_1 and F_2 as vector functions of the angle θ formed between $\hat{r}$ and the x-axis (see Figure 6). Without loss of generality, we assume the taxi-ellipse with center at the origin and F_2 in the first quadrant. For $0 \leq \theta \leq \frac{\pi}{2}$, we have

$$\cos(\theta) = \frac{x_2}{l} \quad \text{and} \quad \sin(\theta) = \frac{y_2}{l},$$

where $l = \dfrac{d_E(F_1, F_2)}{2}$ and d_E is the Euclidean metric. Therefore,

$$F_1 = (-l\cos(\theta), -l\sin(\theta)) \quad \text{and} \quad F_2 = (l\cos(\theta), l\sin(\theta)). \tag{4}$$

Replacing the coordinates of F_1 and F_2 in the equation (3), the relation that determines the points $(x, y) \in \Phi_{F_1,F_2}$ is given by

$$|x + l\cos(\theta)| + |y + l\sin(\theta)| + |x - l\cos(\theta)| + |y - l\sin(\theta)| = 2a. \tag{5}$$

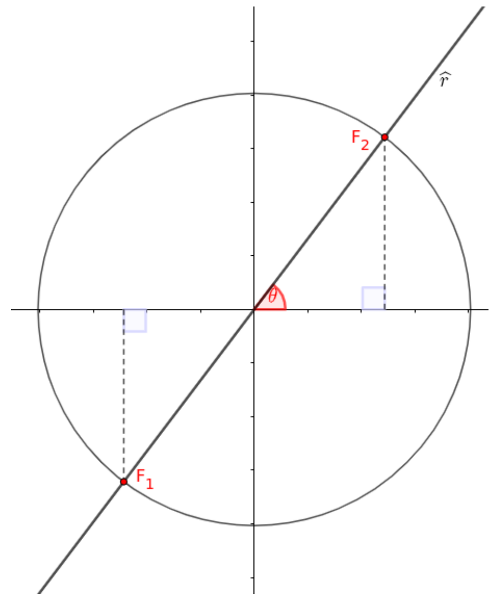

Figure 6: Angle between $\hat{r}$ and the x-axis.

We now rotate the taxi-ellipse Φ_{F_1,F_2} by varying the value of θ in order to analyze the changes that can occur in its geometric configuration. For this, we rotate the focal line $\hat{r}$ determined by F_1 and F_2.

Considering $\theta = 0$ (foci on the x-axis), we have $F_1 = (-l, 0)$ e $F_2 = (l, 0)$. In this case, by using the equations of the line segments r, s, t, r', s', t', u and u' obtained in Section 2 for $x_1 = -x_2 = -l$ and $y_1 = y_2 = 0$, we obtain

$$\begin{aligned}
r &: \; y = -x - a, \text{ for } x \leq -l \text{ and } y \leq 0, \\
s &: \; x = -a, \text{ for } x \leq -l \text{ and } y = 0, \\
t &: \; y = -a + l, \text{ for } -l \leq x \leq l \text{ and } y \leq 0, \\
r' &: \; y = x - a, \text{ for } x \geq l \text{ and } y \leq 0, \\
s' &: \; y = x + a, \text{ for } x \leq -l \text{ and } y \geq 0, \\
t' &: \; y = -x + a, \text{ for } x \geq l \text{ and } y \geq 0, \\
u &: \; x = a, \text{ for } x \geq l \text{ and } y = 0, \\
u' &: \; y = a - l, \text{ for } -l \leq x \leq l \text{ and } y \geq 0.
\end{aligned}$$

Therefore, the geometric configuration in this case is characterized by a hexagon, as we show in Figure 7.

Similarly, considering $\theta = \frac{\pi}{4}$ and using the equations of the line segments r, s, t, r', s', t', u and u' obtained in Section 2 for $x_1 = y_1 = -l\frac{\sqrt{2}}{2}$ e $x_2 = y_2 = l\frac{\sqrt{2}}{2}$, we have an octagon as in

Figure 7.

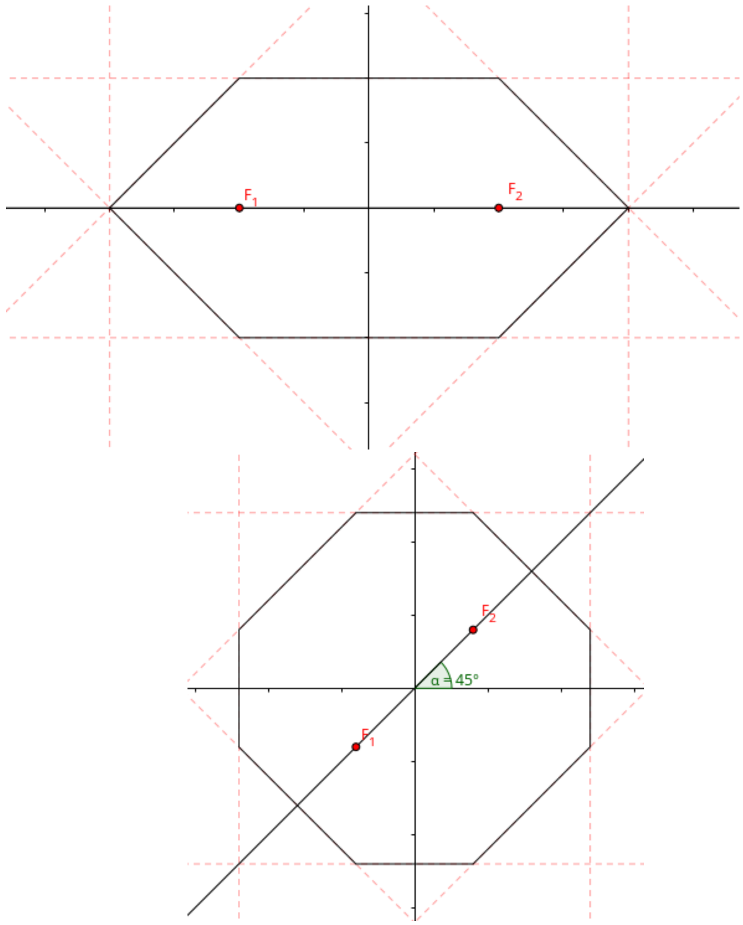

Figure 7: Taxi-ellipse for $\theta = 0$ and $\theta = \frac{\pi}{4}$, respectively.

For $\theta = \frac{\pi}{3}$, taking $x_1 = -x_2 = -\frac{l}{2}$ and $y_1 = -y_2 = -l\frac{\sqrt{3}}{2}$ in the equations of the line segments r, s, t, r', s', t', u and u' obtained in Section 2, we obtain an octagon as in Figure 8.

The last analyzed case refers to $\theta = \frac{\pi}{2}$ (foci on the y-axis) whose geometric configuration is given by a hexagon, as in Figure 8.

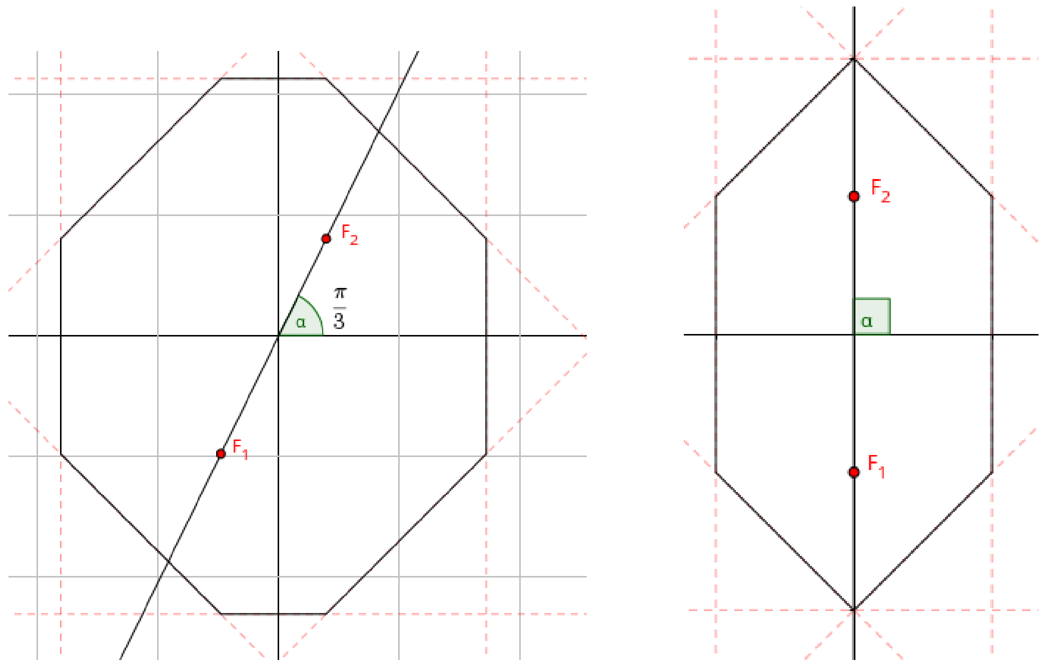

Figure 8: Taxi-ellipse for $\theta = \frac{\pi}{3}$ and $\theta = \frac{\pi}{2}$, respectively.

According to these examples, rotations of angles $\frac{\pi}{4}$, $\frac{\pi}{3}$ and $\frac{\pi}{2}$ clearly change the geometric configuration of a taxi-ellipse and, therefore, are not symmetries. In particular, the rotation $R_{\frac{\pi}{2}}$ is an isometry on $\mathbb{R}^2$ (Proposition 3.4) that is not a symmetry of Φ_{F_1,F_2}.

4. Equation for the regular octagon

The purpose of this section is to obtain an algebraic equation for the regular octagon in the plane. We first explore some properties of the internal angles of the taxi-ellipse Φ_{F_1,F_2} and then show that every regular octagon is a taxi-ellipse.

We restrict our study to the first quadrant, since Φ_{F_1,F_2} is symmetrical with respect to the coordinate axes under the Euclidean metric (see Proposition 2.2). Again, we write F_1 and F_2 as in (4), with $0 \leq \theta \leq \frac{\pi}{2}$. By the fundamental trigonometric identity, the equation (5) can be rewritten as

$$|x + l\cos\theta| + |y + l\sqrt{1-\cos^2\theta}| + |x - l\cos\theta| + |y - l\sqrt{1-\cos^2\theta}| = 2a. \tag{6}$$

Note that $x + l\cos\theta \geq 0$ and $y + l\sqrt{1-\cos^2\theta} \geq 0$, since $\cos\theta \in [0,1]$. Therefore, (6) becomes

$$x + l\cos\theta + y + l\sqrt{1-\cos^2\theta} + |x - l\cos\theta| + |y - l\sqrt{1-\cos^2\theta}| = 2a$$

whose equality implies in the following cases:

a) If $x \geq l\cos\theta$ and $y \geq l\sqrt{1-\cos^2\theta}$, then $(x, y) \in \Phi_{F_1,F_2}$ satisfies the equation $x+y=a$.

b) If $x \geq l\cos\theta$ and $y \leq l\sqrt{1-\cos^2\theta}$, then $(x, y) \in \Phi_{F_1,F_2}$ satisfies the equation $x = a - l\sqrt{1-\cos^2\theta}$.

c) If $x \leq l\cos\theta$ and $y \geq l\sqrt{1-\cos^2\theta}$, then $(x, y) \in \Phi_{F_1,F_2}$ satisfies the equation $y = a - l\cos\theta$.

It is not necessary to consider the case $x \leq l\cos\theta$ and $y \leq l\sqrt{1-\cos^2\theta}$, because there are no points of Φ_{F_1,F_2} in this region. In fact, the symmetric properties of Φ_{F_1,F_2} allow it to be centered at the origin (see Figure 9).

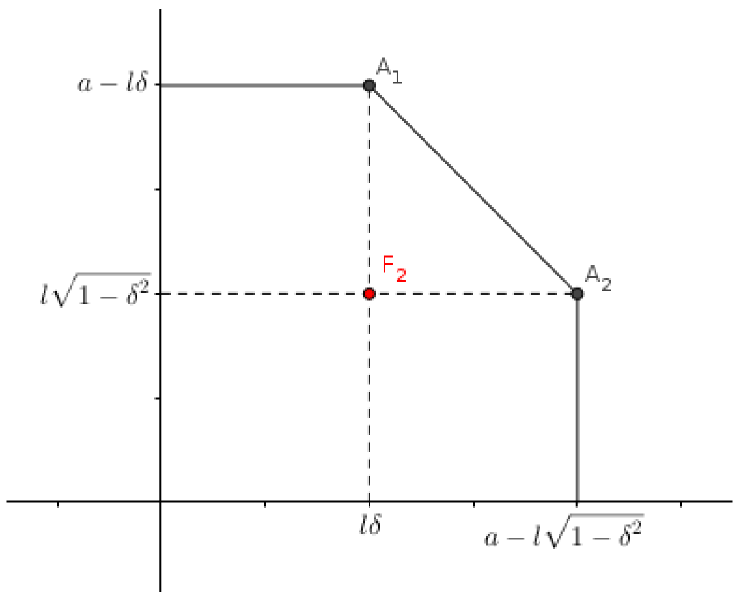

Figure 9: Cases a) − c) for $\delta = \cos\theta$.

First, we consider case a). If $x = l\cos\theta$, then $y = a - l\cos\theta$; if $y = l\sqrt{1-\cos^2\theta}$, then $x = a - l\sqrt{1-\cos^2\theta}$. Denoting $A_1 = (l\cos\theta, a - l\cos\theta)$ and $A_2 = (a - l\sqrt{1-\cos^2\theta}, l\sqrt{1-\cos^2\theta})$, the length C_1 of the line segment $\overline{A_1A_2}$ is given by

$$
\begin{aligned}
C_1 &= \sqrt{(l\cos\theta - (a - l\sqrt{1-\cos^2\theta}))^2 + (a - l\cos\theta - (l\sqrt{1-\cos^2\theta}))^2} \\
&= \sqrt{(l(\cos\theta + \sqrt{1-\cos^2\theta}) - a)^2 + (l(\cos\theta + \sqrt{1-\cos^2\theta}) - a)^2} \\
&= \sqrt{2}|l(\cos\theta + \sqrt{1-\cos^2\theta}) - a| \\
&= \sqrt{2}(a - l(\cos\theta + \sqrt{1-\cos^2\theta})).
\end{aligned}
$$

The last equality follows because $a > c = \frac{1}{2}d_T(F_1, F_2)$, with $F_1 = (-l\cos\theta, -l\sqrt{1-\cos^2\theta})$ e $F_2 = (l\cos\theta, l\sqrt{1-\cos^2\theta})$. Thus, as we consider the first quadrant, we obtain

$$a > \frac{1}{2}\left(|l\cos\theta + l\cos\theta| + |l\sqrt{1-\cos^2\theta} + l\sqrt{1-\cos^2\theta}|\right) = l(\cos\theta + \sqrt{1-\cos^2\theta}),$$

which also implies that $C_1 \neq 0$. Therefore,

$$C_1 = \sqrt{2}(a - l(\cos\theta + \sqrt{1-\cos^2\theta})) > 0. \tag{7}$$

For case b), the points $(x, y) \in \Phi_{F_1,F_2}$ satisfy the equation $x = a - l\sqrt{1-\cos^2\theta}$. In this case, we calculate the length C_2 of the line segment $\overline{A_2B_2}$, where $B_2 = (a - l\sqrt{1-\cos^2\theta}, 0)$ belongs to the x-axis (see Figure 10). Therefore,

$$C_2 = l\sqrt{1-\cos^2\theta}. \tag{8}$$

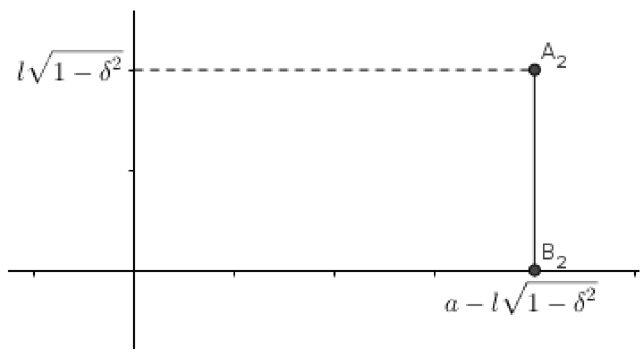

Figure 10: Line segment $\overline{A_2B_2}$ for $\delta = \cos\theta$.

In case c), the points $(x, y) \in \Phi_{F_1,F_2}$ satisfy the equation $y = a - l\cos\theta$. In this case, we calculate the length C_3 of the line segment $\overline{D_1A_1}$, where $D_1 = (0, a - l\cos\theta)$ belongs to the y-axis (see Figure 11). Thus, we obtain

$$C_3 = l\cos\theta. \tag{9}$$

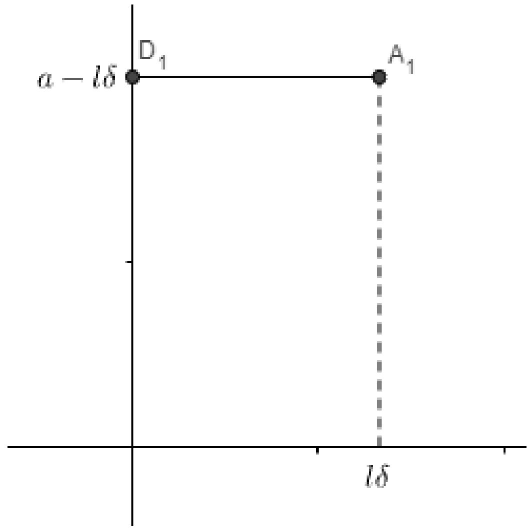

Figure 11: Line segment $\overline{D_1A_1}$ for $\delta = \cos\theta$.

We remark that the lengths C_1, C_2 and C_3 are also determined in (CRUZ, 2015, pp. 49-55) for the case $c = 1$ and $a = 2$. The values obtained in (7), (8) and (9) generalize this case. Based on these values, we now analyze the internal angles of the taxi-ellipse Φ_{F_1,F_2}. The following result allows us to obtain fundamental properties for the construction of the algebraic expression that characterizes the regular octagon.

Lemma 4.1 *Every taxi-ellipse has at least one internal angle equal to $\frac{3\pi}{4}$. Furthermore, all internal angles are equal if and only if C_2 and C_3 are non-zero.*

Proof: We have that C_1 is strictly positive, so $A_1 \neq A_2$. In addition, C_2 and C_3 do not vanish simultaneously. Clearly, the only relevant internal angles for our analysis are those represented in Figure 12 because the others are congruent to them, by Proposition 3.

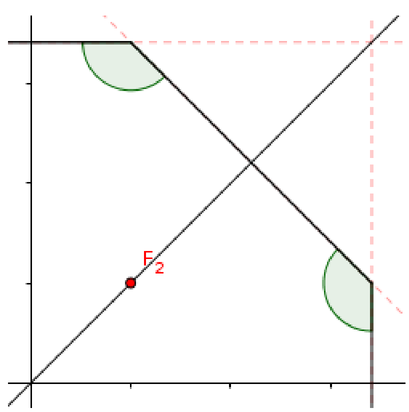

Figure 12: Internal angles.

As we restrict our study to the first quadrant, we compare the angles between the following three line segments:

i) k: line segment of equation $x + y = a$, with $x \in [l\cos\theta, a - l\sqrt{1-\cos^2\theta}]$;

ii) v: vertical line segment, if it exists, of equation $x = a - l\sqrt{1-\cos^2\theta}$, with $x > 0$ and $y \in [0, l\sqrt{1-\cos^2\theta}]$;

iii) h: horizontal line segment, if it exists, of equation $y = a - l\cos\theta$, with $y > 0$ and $x \in [0, l\cos\theta]$.

The non-existence of v and h in the taxi-ellipse Φ_{F_1,F_2} is related to the possibility of C_2 and C_3 being equal to zero, respectively. The segment k always exists, since $C_1 > 0$. In addition, the internal angles of Φ_{F_1,F_2} are determined by the largest angle between the straight lines containing such segments. We divide the proof into three cases:

1. If $C_2 \neq 0$ and $C_3 \neq 0$, then the segments v and h exist, whose direction vectors have coordinates $(-1, 1)$ and $(1, 0)$, respectively. Therefore, the angle $\alpha \in [\frac{\pi}{2}, \pi]$ between k and h is determined by

$$\cos\alpha = \frac{\langle(-1,1),(1,0)\rangle}{||(-1,1)||||(1,0)||} = -\frac{\sqrt{2}}{2}.$$

Hence $\alpha = \frac{3\pi}{4}$. Similarly, we conclude that the angle between the segments k and v is also $\frac{3\pi}{4}$, since its direction vectors have coordinates $(-1, 1)$ and $(0, -1)$, respectively.

2. If $C_2 = 0$, then the segment h exists (since $C_3 \neq 0$). The angle α between k and h has been calculated in the previous case, that is, $\alpha = \frac{3\pi}{4}$. In addition, as $C_2 = 0$, the segment k has its end point on the x-axis (see Figure 7 for $\theta = 0$). In this case, the angle $\beta \in [0, \frac{\pi}{2}]$ between k and the x-axis is determined by

$$\cos\beta = \frac{\langle(-1,1),(-1,0)\rangle}{||(-1,1)||||(-1,0)||} = \frac{\sqrt{2}}{2},$$

that is, $\beta = \frac{\pi}{4}$. According to Proposition 3, the taxi-ellipse constructed in this case is composed of another line segment k' which is the reflection of k across the x-axis. The angle formed between k and k' is equal to $2\beta = \frac{\pi}{2}$.

3. If $C_3 = 0$, then the segment v exists (since $C_2 \neq 0$). The angle α between k and v has also been calculated in the first case, that is, $\alpha = \frac{3\pi}{4}$. Moreover, as $C_3 = 0$, the segment

k has its initial point on the y-axis (see Figure 8 for $\theta = \frac{\pi}{2}$). Similar to the previous case, the angle $\gamma \in [0, \frac{\pi}{2}]$ between k and the y-axis is determined by

$$\cos\gamma = \frac{\langle(1,-1),(0,-1)\rangle}{||(1,-1)||||(0,-1)||} = \frac{\sqrt{2}}{2},$$

that is, $\gamma = \frac{\pi}{4}$. By Proposition 3, the angle between k and the segment k'', which is the reflection of k with respect to the y-axis, is equal to $2\gamma = \frac{\pi}{2}$.

Therefore, from Proposition 3, we conclude the following: in case 1 ($C_2 \neq 0$ and $C_3 \neq 0$), all internal angles of the taxi-ellipse Φ_{F_1,F_2} are equal to $\frac{3\pi}{4}$, which also proves the necessary condition for C_2 and C_3 to be non-zero; in cases 2 and 3 (either $C_2 = 0$ or $C_3 = 0$), at least one internal angle of Φ_{F_1,F_2} is equal to $\frac{3\pi}{4}$. In these cases, every internal angle of Φ_{F_1,F_2} is necessarily equal to $\frac{\pi}{2}$ or $\frac{3\pi}{4}$ and, therefore, Φ_{F_1,F_2} has distinct internal angles. This proves the sufficient condition for C_2 and C_3 to be non-zero.

■

Before the two main results of this section, we will use again the values of C_1, C_2 and C_3 obtained in (7)-(9) to obtain conditions so that a taxi-ellipse Φ_{F_1,F_2} has all its sides equal. For this, we denote by d_E the Euclidean metric and by d_T the taxicab metric. In addition, remember that $\hat{r}$ denotes the line determined by F_1 and F_2 and that $\theta \in [0, \frac{\pi}{2}]$ denotes the angle formed between $\hat{r}$ and the x-axis.

Lemma 4.2 *All sides of a taxicab elipse Φ_{F_1,F_2} are equal if and only if*

$$a = (1+\sqrt{2})l \quad e \quad C_2 = C_3 = \frac{\sqrt{2}}{2}l,$$

where $l = \dfrac{d_E(F_1,F_2)}{2}$. In this case, the angle between the focal line $\hat{r}$ and the x-axis is $\frac{\pi}{4}$.

Proof: Clearly, a taxicab ellipse Φ_{F_1,F_2} has all its sides equal if and only if

$$\frac{C_1}{2} = C_2 = C_3. \tag{10}$$

According to (8) and (9), we have $C_2 = C_3$ if and only if

$$l(\cos\theta - \sin\theta) = 0,$$

which occurs if and only if $\theta = \frac{\pi}{4}$, because $l \neq 0$. In this case

$$C_1 = \sqrt{2}(a - \sqrt{2}l) \quad \text{and} \quad C_2 = C_3 = \frac{\sqrt{2}}{2}l.$$

Thus, the first equality of (10) is valid if and only if $a - \sqrt{2}l = l$, that is[1] , $a = (1 + \sqrt{2})l$.

■

As a consequence, we have the following theorem.

Theorem 4.3 *Every regular octagon is a taxicab ellipse.*

Proof: Let $\mathcal{O}$ be a regular octagon with sides measuring $\tilde{l} > 0$. Consider the circle $\mathcal{C}$ of center at the origin and radius $\frac{\sqrt{2}}{2}\tilde{l}$ under the Euclidean metric. Note that the intersection of $\mathcal{C}$ with the line $y = x$ occurs at the points $\tilde{F}_1 = (-\frac{\tilde{l}}{2}, -\frac{\tilde{l}}{2})$ and $\tilde{F}_2 = (\frac{\tilde{l}}{2}, \frac{\tilde{l}}{2})$. Thus, the line $\hat{r}$ determined by $\tilde{F}_1$ and $\tilde{F}_2$ has equation $y = x$ and forms an angle equal to $\frac{\pi}{4}$ with the x-axis. In addition,

$$d_E(\tilde{F}_1, \tilde{F}_2) = \sqrt{2}\tilde{l}.$$

Consider $\tilde{F}_1$ and $\tilde{F}_2$ as the foci of the taxi-ellipse $\Phi_{\tilde{F}_1,\tilde{F}_2}$. Since the angle θ between $\hat{r}$ and the x-axis is equal to $\frac{\pi}{4}$, it follows from (8) and (9) that

$$C_2 = C_3 = \frac{\sqrt{2}\tilde{l}}{2}\frac{\sqrt{2}}{2} = \frac{\tilde{l}}{2}.$$

Therefore, taking

$$a = (1 + \sqrt{2})\frac{\sqrt{2}\tilde{l}}{2} = (\frac{\sqrt{2}}{2} + 1)\tilde{l},$$

we have by Lemma 4.2 that all sides of $\Phi_{\tilde{F}_1,\tilde{F}_2}$ are equal, namely equal to $\tilde{l}$. Since $C_2 \neq 0$ and $C_3 \neq 0$, it follows from Lemma 4.1 that all the internal angles of $\Phi_{\tilde{F}_1,\tilde{F}_2}$ are equal to $\frac{3\pi}{4}$, that is, $\Phi_{\tilde{F}_1,\tilde{F}_2}$ is a regular octagon with sides measuring $\tilde{l}$. Therefore, $\mathcal{O} = \Phi_{\tilde{F}_1,\tilde{F}_2}$.

■

Based on the previous theorem, we obtain an algebraic equation for the regular octagon with sides measuring $\tilde{l}$.

Corollary 4.4 *Let $\mathcal{O}$ be the regular octagon centered on the origin with sides measuring $\tilde{l}$. Then every point $(x, y) \in \mathcal{O}$ satisfies the equation*

$$|x + \frac{\tilde{l}}{2}| + |y + \frac{\tilde{l}}{2}| + |x - \frac{\tilde{l}}{2}| + |y - \frac{\tilde{l}}{2}| = (\sqrt{2} + 2)\tilde{l}.$$

Proof: By the proof of Theorem 4.3, $\mathcal{O}$ is a taxi-ellipse with foci $\tilde{F}_1 = (-\frac{\tilde{l}}{2}, -\frac{\tilde{l}}{2})$ and $\tilde{F}_2 = (\frac{\tilde{l}}{2}, \frac{\tilde{l}}{2})$ and with $a = (\frac{\sqrt{2}}{2} + 1)\tilde{l}$. Then we use the equation (3).

■

[1] Note that $a > c$, as required in Definition 2.1. In fact, it follows from (LIMA, 2014, p. 6) that $c = \frac{1}{2}d_T(F_1, F_2) \leq d_E(F_1, F_2) = 2l$. So $a = (1 + \sqrt{2})l > 2l \geq c$.

It is natural now to ask whether there are other regular polygons that are taxi-ellipses. The answer to this question is negative, as the following theorem shows.

Theorem 4.5 *The only regular polygons that are taxi-ellipses are regular octagons.*

Proof: By Lemma 4.1, every taxi-ellipse has at least one internal angle equal to $\frac{3\pi}{4}$. As every regular polygon has its internal angles equal and the only ones with internal angles equal to $\frac{3\pi}{4}$ are octagons, the result follows.

■

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