Cálculo Diferencial e Integral: um KIT de sobrevivência

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CURVAS - Triedo móvel de Frenet

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Baseado no artigo:

Visualizing the Frenet-Serret equations D.A. Aruliah

MapleTech, vol. 4, no. 1, pp. 105-110, 1997

e na worksheet de Douglas B. Meade

Para uma curva parametrizada cujo vetor tangente é unitário T(s), valem as seguintes relações conhecidas como as fómulas de Frenet-Serret.

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}, \quad \frac{d\mathbf{N}}{ds} = -\kappa\mathbf{T} - \tau\mathbf{B}, \quad \frac{d\mathbf{B}}{ds} = \tau\mathbf{N}.$$

Onde κ é a curvatura e τ é a torsão.

Pelo Teorema Fundamental da teoria das curvas, curvatura κ e torsão τ determinam (a menos de um movimento rígido no espaço) uma única curva parametrizada.

Esta worksheet determina os vetores [T,N,B], a curvatura e a torsão.

Introdução

Como funciona este procedimento?

Você entra com a definição paramétrica do caminho

Os parâmetros de Frenet são os vetores tangente unitário (T), normal principal (N), e o

binormal (B), a curvatura (k), e a torsão (t).

O primeiro argumento do procedimento FrenetSerretCalc é a definição do caminho e o segundo argumento é o parâmetro.

O comando FrenetSerretPlot usa a forma paramétrica do caminho, os vetores unitários [T , N , B], a curvatura (k), a variação do parâmetro, e o número de frames para criar uma animação do movimento de uma partícula viajando ao longo do caminho especificado. Nesta animação é mostrado os vetores tangente, normal e binormal e o círculo osculador, isto é, o círculo de raio 1/ k que está no plano de movimento em cada ponto dado sobre a curva.

Execute os comandos e faça os exemplos.

Comandos

> **restart:**

> **with(plots):**

> **with(linalg):**

Warning, new definition for norm

Warning, new definition for trace

> **libname := `c:/maplev4/local/frenetserret`, libname;**

libname := c:/maplev4/local/frenetserret, C:\MAPLEV4/update, C:\MAPLEV4/lib

> **with(FrenetSerret); # loads the package**

[*FrenetSerretCalc*, *FrenetSerretPlot*]

> **eval(FrenetSerretCalc): # echoes the procedure**

> **eval(FrenetSerretPlot): # ...likewise**

> **macro(fs=FrenetSerret):**

> **with(fs);**

$$[FrenetSerretCalc, FrenetSerretPlot]$$

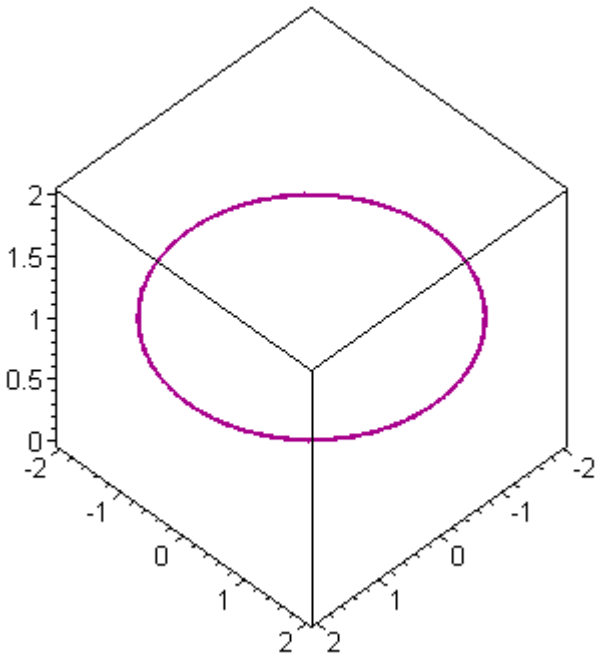
Exemplos

Exemplo 0

> **R := [2*cos(t), 2*sin(t), 1];**

$$R := [2\cos(t), 2\sin(t), 1]$$

> **spacecurve(R,t=-2*Pi..2*Pi,axes=boxed,shading=z,thickness=2);**



> **parameters := FrenetSerretCalc(R,t):**

$$Frenet\ parameters\ calculated:$$

$$[[T, N, B], \kappa, \tau]$$

> **combine(parameters[1], trig):**

> **map(simplify, '', sqrt):**

> **triad := map(normal, '');**

$$triad := [[-\sin(t), \cos(t), 0], [-\cos(t), -\sin(t), 0], [0, 0, 1]]$$

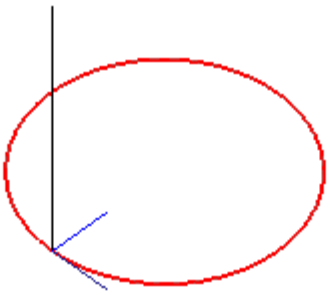
> combine(parameters[2], trig):

> kappa := simplify('', sqrt);

$$\kappa := \frac{1}{2}$$

> movie := FrenetSerretPlot(R, triad, kappa, 0..2*Pi, 10):

> display(movie, insequence=true);

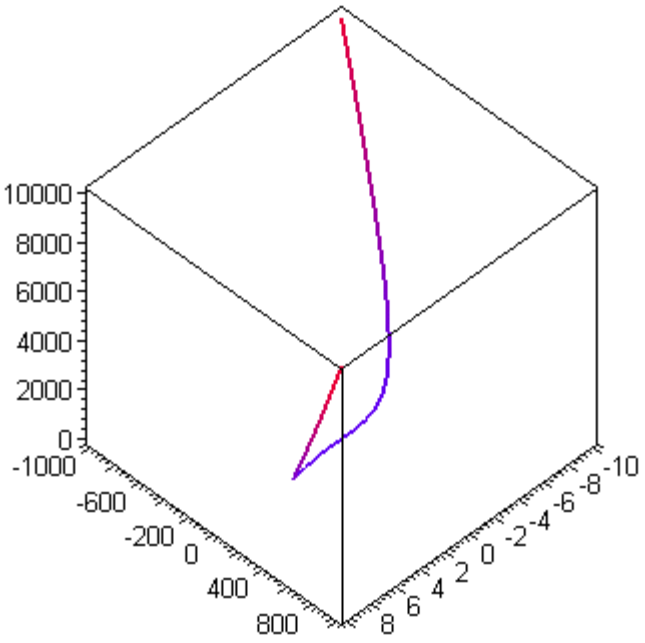


Exemplo 1

> R:=[t,t^3,t^4];

$$R := [t, t^3, t^4]$$

> spacecurve(R,t=-10..10,axes=boxed,shading=z,thickness=2);



> **parameters := FrenetSerretCalc(R,t):**

Frenet parameters calculated:

$$[[T, N, B], \kappa, \tau]$$

> **combine(parameters[1], trig):**

> **map(simplify, '', sqrt):**

> **triad := map(normal, '');**

$$triad := \left[\left[\frac{1}{\sqrt{1+9t^4+16t^6}}, 3\frac{t^2}{\sqrt{1+9t^4+16t^6}}, 4\frac{t^3}{\sqrt{1+9t^4+16t^6}}\right], \left[-\frac{\mathrm{signum}(t)\,t^2\,(3+8t^2)}{\sqrt{4t^6+4t^2+1}\sqrt{1+9t^4+16t^6}}, -\frac{\mathrm{signum}(t)\,(-1+8t^6)}{\sqrt{4t^6+4t^2+1}\sqrt{1+9t^4+16t^6}}, 2\frac{t\,\mathrm{signum}(t)\,(3t^4+1)}{\sqrt{4t^6+4t^2+1}\sqrt{1+9t^4+16t^6}}\right], \left[2\frac{t^3\,\mathrm{signum}(t)}{\sqrt{4t^6+4t^2+1}}, -2\frac{t\,\mathrm{signum}(t)}{\sqrt{4t^6+4t^2+1}}, \frac{\mathrm{signum}(t)}{\sqrt{4t^6+4t^2+1}}\right]\right]$$

> combine(parameters[2], trig):

> kappa := simplify('', sqrt);

>

$$\kappa := 6\,\frac{\mathrm{signum}(t)\,t\sqrt{4\,t^6+4\,t^2+1}}{(1+9\,t^4+16\,t^6)^{3/2}}$$

> movie := FrenetSerretPlot(R, triad, kappa, -1..1, 10):

> display(movie, insequence=true);

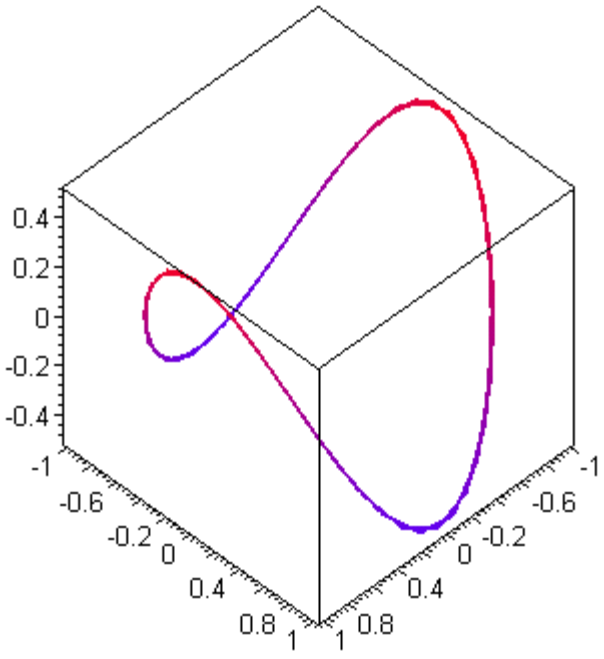
Exemplo 2

> R := [cos(t), sin(t), 1/2*(cos(t)^2-sin(t)^2)];

$$R := \left[\cos(t),\ \sin(t),\ \frac{1}{2}\cos(t)^2-\frac{1}{2}\sin(t)^2\right]$$

> spacecurve(R,t=-2*Pi..2*Pi,axes=boxed,shading=z,thickness=2);

>



> **parameters := FrenetSerretCalc(R,t):**

Frenet parameters calculated:

$$[[T, N, B], \kappa, \tau]$$

> **combine(parameters[1], trig):**

> **map(simplify,", sqrt):**

> **triad := map(normal, ");**

>

$$triad := \left[\left[-2\frac{\sin(t)}{\sqrt{6-2\cos(4t)}}, 2\frac{\cos(t)}{\sqrt{6-2\cos(4t)}}, -2\frac{\sin(2t)}{\sqrt{6-2\cos(4t)}}\right], \left[-\frac{6\cos(t)-3\cos(3t)+\cos(5t)}{\sqrt{6\cos(4t)+14}\sqrt{6-2\cos(4t)}}, -\frac{6\sin(t)+\sin(5t)+3\sin(3t)}{\sqrt{6\cos(4t)+14}\sqrt{6-2\cos(4t)}}, -8\frac{\cos(2t)}{\sqrt{6\cos(4t)+14}\sqrt{6-2\cos(4t)}}\right], \left[-\frac{\cos(3t)+3\cos(t)}{\sqrt{6\cos(4t)+14}}, -\frac{-3\sin(t)+\sin(3t)}{\sqrt{6\cos(4t)+14}}, \frac{2}{\sqrt{6\cos(4t)}}\right]\right]$$

> **combine(parameters[2], trig):**

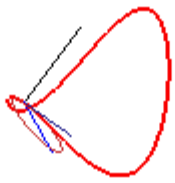
> **kappa := simplify('', sqrt);**

$$\kappa := -2\,\frac{\sqrt{3\cos(4t)+7}}{\sqrt{3-\cos(4t)}\,(-3+\cos(4t))}$$

>

> **movie := FrenetSerretPlot(R, triad, kappa, 0..2*Pi, 10):**

> **display(movie, insequence=true);**



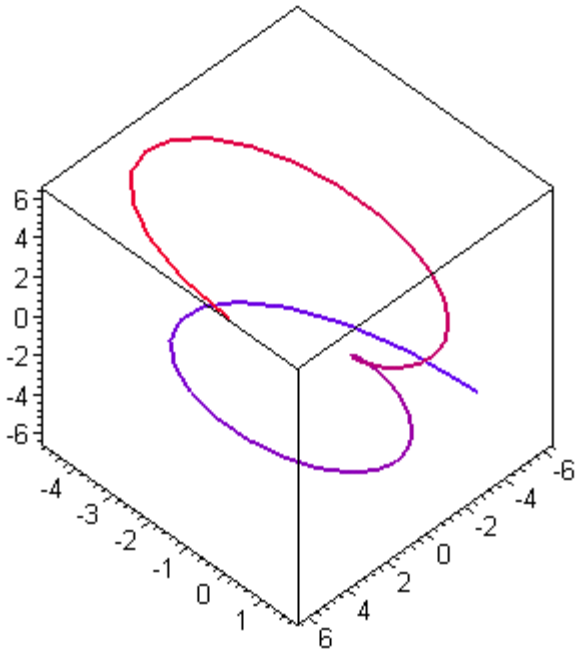
Exemplo 3

> **R := [t*cos(t), t*sin(t), t];**

$$R := [t\cos(t), t\sin(t), t]$$

> **spacecurve(R,t=-2*Pi..2*Pi,axes=boxed,shading=z,thickness=2);**

>



> **parameters := FrenetSerretCalc(R,t);**

Frenet parameters calculated:

$$[[T, N, B], \kappa, \tau]$$

$$parameters := \left[\left[\left[-\frac{-\cos(t)+t\sin(t)}{\sqrt{2+t^2}}, \frac{\sin(t)+t\cos(t)}{\sqrt{2+t^2}}, \frac{1}{\sqrt{2+t^2}}\right], \left[-\frac{4\sin(t)+3t\cos(t)+\sin(t)t^2+t^3\cos(t)}{\sqrt{5t^2+8+t^4}\sqrt{2+t^2}}, -\frac{-4\cos(t)-\cos(t)t^2+3t\sin(t)+t^3\sin(t)}{\sqrt{5t^2+8+t^4}\sqrt{2+t^2}}, -\frac{t}{\sqrt{5t^2+8+t^4}\sqrt{2+t^2}}\right], \left[\frac{-2\cos(t)+t\sin(t)}{\sqrt{5t^2+8+t^4}}, -\frac{2\sin(t)+t\cos(t)}{\sqrt{5t^2+8+t^4}}, \frac{2+t^2}{\sqrt{5t^2+8+t^4}}\right]\right], (4\,c$$

$$+t^2\sin(t)^2+4\sin(t)^2+t^2\cos(t)^2+4\cos(t)^4+8\cos(t)^2t^2\sin(t)^2+8\cos(t)^2\sin(t)^2+4\cos$$

$$+t^4\sin(t)^4+4t^2\sin(t)^4+2t^4\sin(t)^2\cos(t)^2+4\sin(t)^4+t^4\cos(t)^4)^{1/2} \Big/$$

$$(1+\cos(t)^2+t^2\sin(t)^2+\sin(t)^2+t^2\cos(t)^2)^{3/2}, \frac{t^2+6}{5t^2+8+t^4}\Bigg]$$

> combine(parameters[1], trig);

$$\left[\left[\frac{\cos(t)-t\sin(t)}{\sqrt{2+t^2}},\frac{\sin(t)+t\cos(t)}{\sqrt{2+t^2}},\frac{1}{\sqrt{2+t^2}}\right],\left[\frac{-4\sin(t)-3t\cos(t)-\sin(t)t^2-t^3\cos(t)}{\sqrt{5t^2+8+t^4}\sqrt{2+t^2}},\right.\right.$$

$$\left.\frac{4\cos(t)+\cos(t)t^2-3t\sin(t)-t^3\sin(t)}{\sqrt{5t^2+8+t^4}\sqrt{2+t^2}},-\frac{t}{\sqrt{5t^2+8+t^4}\sqrt{2+t^2}}\right],$$

$$\left.\left[\frac{-2\cos(t)+t\sin(t)}{\sqrt{5t^2+8+t^4}},\frac{-2\sin(t)-t\cos(t)}{\sqrt{5t^2+8+t^4}},\frac{2+t^2}{\sqrt{5t^2+8+t^4}}\right]\right]$$

> map(simplify, '', sqrt);

$$\left[\left[\frac{\cos(t)-t\sin(t)}{\sqrt{2+t^2}},\frac{\sin(t)+t\cos(t)}{\sqrt{2+t^2}},\frac{1}{\sqrt{2+t^2}}\right],\left[-\frac{4\sin(t)+3t\cos(t)+\sin(t)t^2+t^3\cos(t)}{\sqrt{5t^2+8+t^4}\sqrt{2+t^2}},\right.\right.$$

$$\left.-\frac{-4\cos(t)-\cos(t)t^2+3t\sin(t)+t^3\sin(t)}{\sqrt{5t^2+8+t^4}\sqrt{2+t^2}},-\frac{t}{\sqrt{5t^2+8+t^4}\sqrt{2+t^2}}\right],$$

$$\left.\left[\frac{-2\cos(t)+t\sin(t)}{\sqrt{5t^2+8+t^4}},-\frac{2\sin(t)+t\cos(t)}{\sqrt{5t^2+8+t^4}},\frac{2+t^2}{\sqrt{5t^2+8+t^4}}\right]\right]$$

> triad := map(normal, '');

$$triad := \left[\left[\frac{\cos(t)-t\sin(t)}{\sqrt{2+t^2}},\frac{\sin(t)+t\cos(t)}{\sqrt{2+t^2}},\frac{1}{\sqrt{2+t^2}}\right],\left[-\frac{4\sin(t)+3t\cos(t)+\sin(t)t^2+t^3}{\sqrt{5t^2+8+t^4}\sqrt{2+t^2}}\right.\right.$$

$$\left.-\frac{-4\cos(t)-\cos(t)t^2+3t\sin(t)+t^3\sin(t)}{\sqrt{5t^2+8+t^4}\sqrt{2+t^2}},-\frac{t}{\sqrt{5t^2+8+t^4}\sqrt{2+t^2}}\right],$$

$$\left.\left[\frac{-2\cos(t)+t\sin(t)}{\sqrt{5t^2+8+t^4}},-\frac{2\sin(t)+t\cos(t)}{\sqrt{5t^2+8+t^4}},\frac{2+t^2}{\sqrt{5t^2+8+t^4}}\right]\right]$$

> combine(parameters[2], trig);

$$\frac{\sqrt{5t^2+8+t^4}}{(2+t^2)^{3/2}}$$

> **kappa := simplify('', sqrt);**

$$\kappa := \frac{\sqrt{5t^2+8+t^4}}{(2+t^2)^{3/2}}$$

> **tau := combine(parameters[3], trig);**

$$\tau := \frac{t^2+6}{5t^2+8+t^4}$$

> **movie := FrenetSerretPlot(R, triad, kappa, 0..2*Pi, 10):**

> **plots[display](movie, insequence=true);**

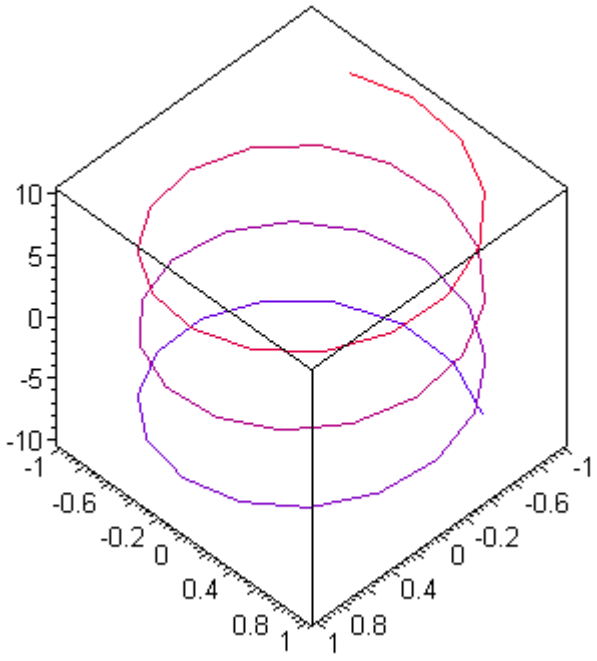
>

Exemplo 4 - helicóide

> **R := [cos(t), sin(t), t];**

$$R := [\cos(t), \sin(t), t]$$

> **spacecurve(R,t=-10..10,axes=boxed,shading=z,thickness=1);**



> **parameters := FrenetSerretCalc(R,t):**

Frenet parameters calculated:

$$[[T, N, B], \kappa, \tau]$$

> **combine(parameters[1], trig):**

> **map(simplify, '', sqrt):**

> **triad := map(normal, '');**

$$triad := \left[\left[-\frac{1}{2}\sin(t)\sqrt{2}, \frac{1}{2}\cos(t)\sqrt{2}, \frac{1}{2}\sqrt{2}\right], [-\cos(t), -\sin(t), 0], \left[\frac{1}{2}\sin(t)\sqrt{2}, -\frac{1}{2}\cos(t)\sqrt{2}, \frac{1}{2}\sqrt{\right.\right.$$

> **combine(parameters[2], trig):**

> **kappa := simplify('', sqrt);**

$$\kappa := \frac{1}{2}$$

>

> **movie := FrenetSerretPlot(R, triad, kappa, 0..10*Pi, 10):**

> **display(movie, insequence=true);**